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Analytical approach to traffic jams

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Abstract. The traffic-flow problem in two dimensions is formulated as a three-state model on a square lattice in terms of Pauli operators. Using a Fock-space representation of the master equation we get the Liouvillian for the problem with asymmetric exclusion. Three different realizations, symmetric, right-before-left model and an exchange model will be analysed within the mean-field approximation (MFA). The resulting kinetic equations for the average occupation number of cars in the upward and sideward directions are coupled. The average velocity can be calculated in MFA. It results in a jamming transition that depends on the total concentration of cars.

Traffic problems have been studied in recent years using the availability of powerful supercomputers. Such traffic simulations based on hydrodynamic models [1–4] have provided much insight and are in good agreement with experiments for simple systems. However, the simulation of the underlying collective behaviour of the traffic flow in a whole city requires a many body description for classical systems with a stochastic dynamics.

Biham *et al* [6] have applied a three-state cellular automaton model to simulate a simple model that describes traffic flow in two dimensions. In varying the system size from 16×16 to 512×512 particles (cars) they found a critical concentration at which a sharp transition occurs from a phase of freely moving cars to a phase where all cars are stopped. The critical concentration was determined to be $\rho_c \approx 0.31$ for the largest system.

In a series of papers Nagatani [7–14] has studied in detail the traffic flow including the jamming transition proposed originally by Biham *et al* [6]. He confirmed the results given in [6]. Furthermore, he discussed some extensions of the model.

Whereas all the previous studies are based above all on a numerical treatment, here we present an analytical approach using a Fock-space formulation of the problem. Such a formulation was developed originally in terms of Bose operators to simulate the dynamical behaviour of the system starting with the master equation [18–20]. An extension to systems with exclusion was proposed recently in [22] where any double occupancy at a certain lattice point is excluded.

There are also other papers in which the problem is discussed with various methods [15–17] different to ours.

To be specific we discretize the system by introducing a square lattice in two dimensions. Each of the lattice sites is in either one of three states: containing a car driving from left to right, denoted by A, a car driving upwards (D), or being empty (E). The set of the states of the whole lattice are characterized by the state vector n, where $n_{i\alpha}$ gives the state of site

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i, α of the lattice. Here, i counts the sites from the bottom to the top of the lattice whereas α numbers the lattice sites from left to right. Periodic boundary conditions are assumed.

We remark here that there is a similar realization by a three-candidate voter model discussed in [5].

In a first version the dynamics of the system is governed by the following rules. While a D car can jump with the rate ν_1 in vertical direction, an A car jumps with the rate ν_2 in horizontal direction. In contrast to [6], the dynamics is not controlled by a traffic light such that the A cars move only in even time steps and the D cars move only in odd time step. Here, we consider the collective behaviour of simultaneous asymmetric hopping under exclusive conditions, that means any double occupancy of a lattice site is forbidden.

To include the restriction we transform the underlying master equation into an equation in a Fock space where the corresponding evolution operator is expressed by Pauli operators.

The master equation reads

$$\partial_t P(n,t) = L' P(n,t) \tag{1}$$

where P is the probability of a transition from an empty lattice at $t = 0$ to the state characterized by n at time t and L' is an operator specified below.

Following Doi [18] (cf also [22]), the probability distribution $P(n,t)$ can be related to a state vector $|F(t)\rangle$ in a Fock-space according to $P(n,t) = \langle n|F(t)\rangle$ with the basisvectors $|n\rangle$. The master equation can be transformed to an equivalent equation in a Fock space

$$\partial_t |F(t)\rangle = \hat{L}|F(t)\rangle\,. \tag{2}$$

The operator L' in (1) is mapped onto the operator $\hat{L}$ given in a second quantized form with $a, a^\dagger$ and $d, d^\dagger$ being the annihilation and creation operators for the A and D cars, respectively.

Usually $\hat{L}$ is expressed in terms of creation and annhiliation operators which satisfy Bose commutation rules [18–21]. In our problem where cars can jump with a certain probability from lattice site to lattice site one has to take into account the exclusion principle. The method can be extended to the case of restricted occupation numbers per lattice site [22–26]. To preserve in addition the restriction of the occupation number in the underlying dynamical equations, the commutation rules of the operators a and d are those of Pauli operators:

$$\begin{aligned} &[d_{i\alpha}, d^\dagger_{j\beta}] = \delta_{i,j}\delta_{\alpha\beta}(1 - 2d^\dagger_{i\alpha}d_{i\alpha}) \\ &[d_{i\alpha}, d_{j\beta}] = [d^\dagger_{i\alpha}, d^\dagger_{i\alpha}] = 0 \\ &d^2_{i\alpha} = (d^\dagger_{i\alpha})^2 = 0 \end{aligned} \tag{3}$$

and similarly for a and $a^\dagger$. As the operators a and d act in different subspaces, all the commutators between them disappear.

The relation between the quantum formalism and the probability approach based upon the master equation can be found by expanding the vector $|F(T)\rangle$ with respect to the basisvectors of the Fock space

$$|F(t)\rangle = \sum_{n_i} P(n,t)|n\rangle\,. \tag{4}$$

As was shown by Doi [18] the average of a physical quantity $B(n)$ is given by the average of the corresponding operator $\hat{B}(t)$

$$\langle \hat{B}(t)\rangle = \sum_{n_i} P(n,t)B(n) = \langle s|\hat{B}|F(t)\rangle\,. \tag{5}$$

This rule also remains valid in the case of Pauli operators [22] with the notation $\langle s| = \langle 0|\exp\sum_i d_i$ where d_i is an annihilation operator. We remark that the normalization

condition is manisfested in the relation $\langle s|F(t)\rangle = 1$ and the sum-vector $\langle s|$ can be expressed by $\langle s| = \sum\langle n|$. The evolution equation for an operator $\hat{B}$ can be written

$$\partial_t\langle\hat{B}) = \langle s|[\hat{B}, \hat{L}]|F(t)\rangle\,. \tag{6}$$

Here we have used the relation $\langle s|\hat{L} = 0$. The dynamics of the classical problem is determined purely by the commutation rules of the underlying operators.

We consider three kinds of models distinguished only by the form of the evolution operator $\hat{L}$ in (2). In two versions the operator consists of two parts, one for the D cars and the other for the A cars.

$$\hat{L} = L_{\mathrm{D}} + L_{\mathrm{A}}\,. \tag{7}$$

In a last version we take also into consideration the exchange operator L_{AD}, L_{DA}.

As a first realization, called symmetric model, the evolution operator is defined in accordance with the dynamics defined above.

For this situation we get

$$\begin{aligned} L_{\mathrm{D}} &= \nu_1 \sum_{i\alpha}[d^\dagger_{i\alpha}d_{i-1\alpha} - d_{i\alpha}d^\dagger_{i\alpha}d^\dagger_{i-1\alpha}d_{i-1\alpha}]\hat{a}_{i\alpha}a^\dagger_{i\alpha}a_{i\alpha-1}a^\dagger_{i\alpha-1} \\ L_{\mathrm{A}} &= \nu_2 \sum_{i\alpha}[a^\dagger_{i\alpha}a_{i\alpha-1} - a_{i\alpha}a^\dagger_{i\alpha}a^\dagger_{i\alpha-1}a_{i\alpha-1}]\hat{d}_{i\alpha}d^\dagger_{i\alpha}d_{i-1\alpha}d^\dagger_{i-1\alpha}\,. \end{aligned} \tag{8}$$

The terms in brackets make sure that a car of one type can only move if the starting site is indeed occupied and the termination point is empty. The additional terms allow only jumps to sites to which at the same time cars of the other type do not jump.

Using the operator (8) and equations (6), (3) for the average of the particle number operator $D_{i\alpha} = d^\dagger_{i\alpha}d_{i\alpha}$ we get

$$\begin{aligned} \partial_t\langle D_{i\alpha}\rangle = \nu_1[&\langle D_{i-1\alpha}(1 - C_{i\alpha} + D_{i\alpha}A_{i\alpha})(1 - A_{i\alpha-1})\rangle \\ &-\langle D_{i\alpha}(1 - C_{i+1\alpha} + D_{i+1\alpha}A_{i+1\alpha})(1 - A_{i+1\alpha-1})\rangle]\,. \end{aligned} \tag{9}$$

The corresponding equation for the A car reads

$$\begin{aligned} \partial_t\langle A_{i\alpha}\rangle = \nu_2[&\langle A_{i\alpha-1}(1 - C_{i\alpha} + D_{i\alpha} + A_{i\alpha})(1 - D_{i-1\alpha}) \\ &-A_{i\alpha}(1 - C_{i\alpha+1} + D_{i\alpha} + A_{i\alpha})(1 - D_{i-1\alpha+1})\rangle] \end{aligned} \tag{10}$$

where $C_{i\alpha} = D_{i\alpha} + A_{i\alpha}$ is the total particle number operator at a lattice site.

Since the change of the mean local occupation number at one lattice site consists of two parts, namely a local incoming current and a local outgoing current of cars, we define local current densities for each type of car:

$$\begin{aligned} j^{\mathrm{d}}_{i-1\to i\alpha} &= \nu_1\,\langle D_{i-1\alpha}(1 - C_{i\alpha} + D_{i\alpha}A_{i\alpha})(1 - A_{i\alpha-1})\rangle \\ j^{\mathrm{a}}_{i\alpha-1\to\alpha} &= \nu_2\,\langle A_{i\alpha-1}\,(1 - C_{i\alpha} + D_{i\alpha}A_{i\alpha})\,(1 - D_{i-1\alpha})\rangle\,. \end{aligned} \tag{11}$$

In analogy with the relation $\mathbf{j} = \rho\mathbf{v}$ we define local mean velocities

$$\begin{aligned} v^{\mathrm{d}}_{i-1\to i\alpha} &= \frac{j^{d}_{i-1\to i\alpha}}{\langle D_{i-1\alpha}\rangle} \\ v^{\mathrm{a}}_{i\alpha-1\to\alpha} &= \frac{j^{\mathrm{a}}_{i\alpha-1\to\alpha}}{\langle A_{i\alpha-1}\rangle}\,. \end{aligned} \tag{12}$$

All further calculations will be performed in the mean-field limit where the car densities should be normalized separately to unity. We have to average locally over all occupation

configurations at one lattice site, and furthermore over all lattice sites in order to get the global mean velocities. The result is

$$\begin{aligned} v_d &= \nu_1[1-\rho+\rho_a\rho_d][1-\rho_a] \\ v_a &= \nu_2[1-\rho+\rho_d\rho_a][1-\rho_d]\,. \end{aligned} \tag{13}$$

The average velocity $v=\frac{1}{2}(v_a+v_d)$ has the form $v=-c_1\rho^3+c_2\rho^2-c_3\rho+c_4$ with $c_i\geqslant 0$ which is qualitatively in agreement with the shape of the graphs given in [11].

The jamming transition occurs at a critical concentration ρ_c where the mean velocity vanishes

$$\begin{aligned} 0 = {} & \nu_1\left[1-\rho_c(1+f)+\rho_c^2 f^2+\rho_c^3 f^2(1-f)\right] \\ & +\nu_2\left[1-\rho_c(2+f)+\rho_c^2(1-f)^2+\rho_c^3 f(1-f)^2\right] \end{aligned} \tag{14}$$

where $\rho_a=f\rho$. This equation can be solved exactly. Here we consider the limit where one hopping rate is very small compared to the other one. Thus we get

$$\begin{aligned} \rho_c &= \frac{1}{2f}\left[1-\frac{1-2f}{f}\frac{\nu_2}{\nu_1}\right] && \text{if } \tfrac{1}{2}<f<1 \text{ and } \nu_1\gg\nu_2 \\ \rho_c &= \frac{1}{2(1-f)}\left[1+\frac{1-2f}{1-f}\frac{\nu_1}{\nu_2}\right] && \text{if } 0<f<\tfrac{1}{2} \text{ and } \nu_1\ll\nu_2\,. \end{aligned} \tag{15}$$

Let us remark that equations (14), (15) only hold provided both concentrations ρ_a or ρ_d are greater than zero. If one concentration vanishes, the corresponding current and the velocity also has to vanish and the critical concentration becomes unity.

As a further realization we have considered the 'right-before-left' case defined by the evolution operator

$$\begin{aligned} L_D &= \nu_1\sum_{i\alpha}[d^\dagger_{i\alpha}d_{i-1\alpha}-d_{i\alpha}d^\dagger_{i\alpha}d^\dagger_{i-1\alpha}d_{i-1\alpha}]a_{i\alpha}a^\dagger_{i\alpha} \\ L_A &= \nu_2\sum_{i\alpha}[a^\dagger_{i\alpha}a_{i\alpha-1}-a_{i\alpha}a^\dagger_{i\alpha}a^\dagger_{i\alpha-1}a_{i\alpha-1}]d_{i\alpha}d^\dagger_{i\alpha}d_{i-1\alpha}d^\dagger_{i-1\alpha}\,. \end{aligned} \tag{16}$$

This operator favours the D cars at a crossing point. An A-type car has to wait before it can jump to site (i,α).

Using the same approximation as before we get the mean velocities

$$\begin{aligned} v_d &= \nu_1(1-\rho+\rho_a\rho_d) \\ v_a &= \nu_2(1-\rho+\rho_a\rho_d)(1-\rho_d)\,. \end{aligned} \tag{17}$$

The jamming transition occurs at

$$\rho_c^{rl}\simeq\frac{1+\nu_1/\nu_2}{2(1-f)} \qquad \text{if } 0<f<\frac{1}{2}\,. \tag{18}$$

Obviously the critical density ρ_c^{rl} is larger than that in the previous model.

In the last model we avoid double occupancy by introducing an exchange operator through which an A-type car can be transformed to that of a D-type and vice versa. The exchange operators L_{AD} and L_{DA} are defined by

$$\begin{aligned} L_{DA} &= \mu_1\sum_{i\alpha}\left[a^\dagger_{i-1\alpha}d_{i-1\alpha}-d^\dagger_{i-1\alpha}d_{i-1\alpha}\right]a^\dagger_{i\alpha-1}a_{i\alpha-1} \\ L_{AD} &= \mu_2\sum_{i\alpha}\left[d^\dagger_{i\alpha-1}a_{i\alpha-1}-a^\dagger_{i\alpha-1}a_{i\alpha-1}\right]d^\dagger_{i-1\alpha}d_{i-1\alpha}\,. \end{aligned} \tag{19}$$

These evolution operators simulate the situation that whenever a lattice site is already occupied the corresponding cars turn off with the exchange rate μ.

As a result additional terms occur in (13) for the symmetrized model:

$$\begin{aligned} v_d &= v_1[1-\rho+\rho_a\rho_d][1-\rho_a]-\mu_1(1-\rho_a)\rho_a+\mu_2(1-\rho_d)\rho_a \\ v_a &= v_2[1-\rho+\rho_a\rho_d][1-\rho_d]+\mu_1(1-\rho_a)\rho_d-\mu_2(1-\rho_d)\rho_d\,. \end{aligned} \tag{20}$$

In the case of the right-before-left model the first parts of the last equation should be replaced by (17). The corresponding equations can be solved exactly. Here we present an estimation of the average velocity in the limit $\mu/v \ll 1$ with $v_1 = v_2 = v$ and $\mu_1 = \mu_2 = \mu$. The critical concentration is given by

$$\rho_c = \frac{1}{2f}\left[1+\frac{\mu}{v}\right] \qquad \text{if } \frac{1}{2} < f < 1$$

$$\rho_c = \frac{1}{2(1-f)}\left[1+\frac{\mu}{v}\right] \qquad \text{if } 0 < f < \frac{1}{2}\,.$$

In this paper we have considered the simplest realization of two type particle hopping processes on a lattice using an analytical approach. The particles denoted as cars can only jump from lattice sites if the corresponding site is not occupied by any other car. The dynamics of this problem is realized in terms of Pauli operators. Depending on the total car density we find a jamming transition at a critical concentration of cars. The method can also be used to discuss higher-order results or can be considered as the starting point for a path integral representation for the problem discussed here.

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